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A note on the qualitative structure of a quasi-stationary tropical cyclone in the vicinity of the eye

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Abstract. A fully mature tropical cyclone is a complicated phenomenon, in which effects such as boundary-layer friction, latent heat release, and so on, must be accounted for, in order to give a complete description of the flow. Nevertheless, it is demonstrated here that a two-layer compressible atmosphere model, with a sharp interface separating the layers, is capable of giving a non-linear vortex-type solution at the lowest order of approximation. The 'eye' of the cyclone is formed when the interface is drawn down to sea level. Heat energy from the ocean into the cyclone and secondary flow within the vortex are both incorporated. The model provides an approximate description of cyclone behaviour, in terms that are easily understood. More complete descriptions are apparently only possible numerically, however, with a corresponding loss of ease of comprehension.

Key words: atmospheric interface, eye radius, perturbation expansion, vortex.

1. Introduction

Tropical cyclones, also known regionally as hurricanes or typhoons, are the biggest storm systems on earth, and their destructive power, particularly in such densely populated areas as the Bay of Bengal and the Eastern United States, is widely documented (*e.g.* Lutgens and Tarbuck [1]). In Australia, too, their drought-breaking benefits are likewise offset by their potential for great harm, as was made evident on Christmas morning in 1974 when Cyclone Tracy devastated the most northerly capital Darwin with winds of up to 217 km/hr. Earlier that same year, Cyclone Wanda had inflicted a 200 million dollar damage bill on the eastern city of Brisbane [2]. A history of the damage caused by some of Australia's more severe tropical cyclones is to be found in the book by Holthouse [3].

Considerable effort has been expended in the theoretical modelling of cyclone behaviour, with a major aim being the eventual prediction of the storm path. A full review of this literature is beyond the scope of this note, although an important paper in the area is that by Gent and McWilliams [4], in which the stability of barotropic vortices was considered. Previous results were reviewed, and the linear stability of circular vortices modelled with a potential vorticity equation was analyzed. A numerical study of a nonlinear model showed how, at finite amplitude, the circular vortex evolves into an asymmetric shape. Smith [5] considered the effect of a shear flow upon the motion of a barotropic vortex.

A number of authors have also sought to give simple analytical relationships between the pressure in the cyclone and the wind strength, for example, with a view to practical prediction. Atkinson and Holliday [6] suggested a simple power-law relation based on experimental data, and an exponential dependence of the pressure on the radial coordinate is advocated by Holland [7], based on a study of nine Florida hurricanes. More recently, Shapiro and

Montgomery [8] have devised a method of analyzing the flow in an atmospheric vortex, by constructing perturbation series for the flow variables, involving a small parameter which is a ratio of orbital to inertial frequencies.

The present paper is also written in the spirit of these simpler analytical studies. We aim to demonstrate the important qualitative features of a quasi-stationary mature cyclone with a theory that is sufficiently simple as to permit easy comprehension, yet without the need to resort to experimental data fitting. Because the present analysis does not seek to allow for the full complexity of cyclone physics, the predictions it makes will be largely qualitative. Nevertheless, we believe they offer an important insight that is often obscured by the complexity of more complete models which must perforce be solved numerically.

For simplicity, two distinct fluid layers are considered in this model; each is compressible and there is assumed to be a sharp interface separating them. In the upper layer, the temperature will be assumed to be constant and the fluid will be supposed to be stationary, so that the density and pressure decay exponentially with height, according to the usual rules for an isothermal gas. A vortex is present in the lower fluid layer. In order that the analysis be kept as simple as possible, we only consider the case of an axially symmetric vortex in this paper, and Shapiro and Montgomery [8] indicate that strong cyclones do indeed tend to be highly symmetric.

The interface between the two fluid layers is found to play a crucial role in the structure of the cyclone in the present model. As the cyclone develops, the interface becomes drawn down, so that for a steadily rotating mature cyclone, the interface is pulled right down to sea level, where it forms the 'eye' of the storm. In addition to providing an analytical relation between pressure and windspeed in the cyclone, the present model therefore also offers a formula for computing the eye radius of the cyclone. This may prove beneficial in estimating the storm strength from knowledge of the eye radius.

The purpose of the present paper is therefore primarily to determine the location of this interface between the two fluid layers in the atmosphere, in which rotation occurs only in the outer lower fluid layer. Pressures and windspeeds in this layer are discussed.

2. Physical model and governing equations

We consider a novel atmospheric vortex, involving a stationary upper fluid (layer 1) separated by a sharp interface from a lower layer 2. Each fluid is compressible and is assumed to obey the ideal gas law, with universal gas constant R and ratio of specific heats γ. The downward acceleration g of gravity influences the air in each fluid layer, and Coriolis forces due to the earth's rotation are included in the usual way (in the f-plane) as characterized by the Coriolis parameter $f = 2\Omega \sin \phi$, in which Ω is the angular speed of the earth and the angle ϕ is measured from the North Pole.

For simplicity, the vortex is assumed to be axially symmetric, and so a coordinate system is located at the centre of the vortex, with the z-axis pointing vertically and r measuring the radial distance from the centre. The interface between the two layers of air is assumed to have the mathematical form $r = \mathcal{R}(z)$, and far away from the cyclone it is assumed that lower layer 2 has some uniform height H, so that $\mathcal{R}(z) \to \infty$ as $z \to H$. Far from the centre of the cyclone, and at sea level, it is assumed that the air pressure and temperature have the constant values p_{atm} and T_s, respectively. For a mature tropical cyclone, the interface is drawn down to the sea surface level $z = 0$ at some (eye) radius $r = A_E$, and therefore $\mathcal{R}(0) = A_E$.

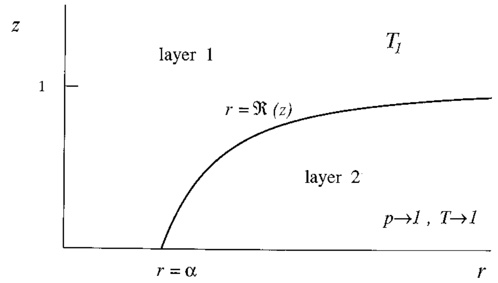

Figure 1. The dimensionless radial and vertical coordinate system. The two air layers and the interface separating them are sketched on the diagram.

It is convenient to introduce dimensionless variables in this analysis, and all lengths have therefore been scaled relative to the asymptotic depth H of lower layer 2. Pressures and temperatures are non-dimensionalized using reference quantities p_{atm} and T_s, and all speeds are scaled in relation to the quantity $\sqrt{\gamma R T_s}$. The unit of density is $p_{\text{atm}}/(RT_s)$. From this point in the analysis, all variables will be assumed to be in non-dimensional form, unless explicitly stated otherwise.

A sketch of the non-dimensional flow situation is given in Figure 1, and it is found that four dimensionless groupings of parameters are involved in the statement of the problem. These are the ratio γ of specific heats of the air, as defined above, a constant $\varepsilon = (\gamma - 1)gH/(\gamma R T_s)$ which is essentially a measure of compressibility, and a dimensionless Coriolis parameter $\kappa = fH/\sqrt{\gamma R T_s}$. In addition, the nondimensional radius of the eye of the cyclone is $\alpha = A_E/H$, and this constant is indicated in Figure 1.

In the upper layer 1 the air is assumed to be stationary. In order to account for heat transfer within the cyclone's eye, it will be assumed that the temperature in layer 1 has a linear profile below some (dimensionless) height $z = \delta$, and becomes constant (isothermal) above this height. Thus the temperature T_1 in layer 1 has the approximate form

$$T_1(z) = \begin{cases} 1 - (1 - T_0)z/\delta & 0 < z < \delta \\ T_0 & z > \delta. \end{cases} \tag{2.1}$$

This form (2.1) for the temperature in the upper fluid layer is consistent with experiment, as indicated by Dutton [9, pp. 83–84]. As the temperature has been made dimensionless with the sea-surface temperature as a reference, then $T_1(0) = 1$ in Equation (2.1), as expected. In the isothermal part of the atmosphere, the dimensional temperature is approximately -70°C, so that for a sea-surface temperature of 30°C, the non-dimensional parameter T_0 in Equation (2.1) would be expected to have a value of about $T_0 = 0{\cdot}67$. Dutton [9] indicates that the atmosphere becomes isothermal at a height of about 11 kms, and the dimensional height of the top of the cyclone is approximately 15 kms (see, for example, Tarakanov [10, p. 163]). Consequently, the non-dimensional parameter δ in Equation (2.1) is expected to have a value of about $\delta = 0{\cdot}73$.

In non-dimensional variables, the ideal gas law has the form $p_1 = \rho_1 T_1$ and, since layer 1 is stationary, it follows that the air density varies with height z according to the formula

$$\rho_1(z) = \begin{cases} A_1 \beta \left[\dfrac{1}{T_0} - \dfrac{1 - T_0}{T_0} \dfrac{z}{\delta} \right]^{\mu - 1} & 0 < z < \delta \\ A_1 \exp\left(-\dfrac{\gamma \varepsilon z}{(\gamma - 1) T_0} \right) & z > \delta, \end{cases} \tag{2.2}$$

where for ease of notation the constants

$$\begin{aligned} \beta &= \exp\left(-\frac{\gamma\varepsilon\delta}{(\gamma-1)T_0}\right), \\ \mu &= \frac{\gamma\varepsilon\delta}{(\gamma-1)(1-T_0)} \end{aligned} \tag{2.3}$$

have been defined. Similarly, the pressure in layer 1 is found to be

$$p_1(z) = \begin{cases} A_1 T_0 \beta \left[\dfrac{1}{T_0} - \dfrac{1-T_0}{T_0}\dfrac{z}{\delta}\right]^{\mu} & 0 < z < \delta \\ A_1 T_0 \exp\left(-\dfrac{\gamma\varepsilon z}{(\gamma-1)T_0}\right) & z > \delta. \end{cases} \tag{2.4}$$

The constant A_1 in Equations (2.2) and (2.4) is yet to be determined.

Suppose that the velocity vector in lower fluid layer 2 has components (u_2, v_2, w_2) in the (r, θ, z) directions, respectively. Conservation of mass requires that

$$\frac{1}{r}\frac{\partial(r\rho_2 u_2)}{\partial r} + \frac{\partial(\rho_2 w_2)}{\partial z} = 0. \tag{2.5}$$

In this equation, as in the following, it has been assumed that there is no dependence on angle θ, so that the vortex is axi-symmetric. In non-dimensional variables, the radial component of the momentum conservation equation becomes (in the neglect of viscous forces)

$$u_2\frac{\partial u_2}{\partial r} + w_2\frac{\partial u_2}{\partial z} - \frac{v_2^2}{r} + \frac{1}{\gamma\rho_2}\frac{\partial p_2}{\partial r} - \kappa v_2 = 0, \tag{2.6}$$

the tangential component gives

$$u_2\frac{\partial v_2}{\partial r} + w_2\frac{\partial v_2}{\partial z} + \frac{u_2 v_2}{r} + \kappa u_2 = 0, \tag{2.7}$$

and the conservation of momentum in the axial direction yields

$$u_2\frac{\partial w_2}{\partial r} + w_2\frac{\partial w_2}{\partial z} + \frac{1}{\gamma\rho_2}\frac{\partial p_2}{\partial z} = -\frac{\varepsilon}{\gamma-1}. \tag{2.8}$$

Energy is also conserved in this layer, and this fact is expressed by the equation

$$u_2\frac{\partial \mathcal{H}_2}{\partial r} + w_2\frac{\partial \mathcal{H}_2}{\partial z} = -\frac{\varepsilon w_2}{\gamma-1}, \tag{2.9a}$$

in which the symbol $\mathcal{H}_2$ denotes the total enthalpy, defined by the expression

$$\mathcal{H}_2 = \frac{p_2}{(\gamma-1)\rho_2} + \tfrac{1}{2}(u_2^2 + v_2^2 + w_2^2) \tag{2.9b}$$

in these dimensionless coordinates.

The dynamic boundary condition to be satisfied on the interface is that pressure should be continuous, as expressed by the relation

$$p_1 = p_2 \quad \text{on } r = \mathcal{R}(z). \tag{2.10}$$

There is also a kinematic condition

$$u_2 = w_2 \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}z} \quad \text{on } r = \mathcal{R}(z), \tag{2.11}$$

which indicates that the fluid in lower layer 2 is not free to cross the interface. If the sea surface is approximated as a rigid plane on $z = 0$, then it also follows that

$$w_2 = 0 \quad \text{on } z = 0. \tag{2.12}$$

Finally, the conditions far from the cyclone are

$$p_2(r, 0) \to 1, \qquad T_2(r, 0) \to 1 \quad \text{as } r \to \infty, \tag{2.13}$$

where the temperature $T_2 = p_2/\rho_2$ in the lower fluid is as defined by the (dimensionless) ideal gas law. Thus the temperature at the sea surface is given, and this is the mechanism by which heat energy is supplied to the cyclone in the present model. In addition, there is a density jump $\Delta\rho$ at the interface, far from the cyclone, which is expressed by the equation

$$\Delta\rho = \rho_2(\infty, 1) - \rho_1(1) > 0. \tag{2.14}$$

3. The perturbation scheme

3.1. The series expansions

After some considerable analysis of the governing Equations (2.1)–(2.14), it becomes evident that an expansion in powers of the Coriolis parameter κ is appropriate near the eye of the storm, and must take the form

$$\begin{aligned}
v_2(r, z) &= v_{2,0}(r) - \tfrac{1}{2}\kappa r + \kappa^2 v_{2,2}(r, z) + \mathcal{O}(\kappa^3), \\
u_2(r, z) &= \kappa u_{2,0}(r, z) + \kappa^3 u_{2,2}(r, z) + \mathcal{O}(\kappa^4), \\
w_2(r, z) &= \kappa w_{2,0}(r, z) + \kappa^3 w_{2,2}(r, z) + \mathcal{O}(\kappa^4), \\
p_2(r, z) &= p_{2,0}(r, z) + \kappa^2 p_{2,2}(r, z) + \mathcal{O}(\kappa^3), \\
\rho_2(r, z) &= \rho_{2,0}(r, z) + \kappa^2 \rho_{2,2}(r, z) + \mathcal{O}(\kappa^3), \\
\mathcal{R}(z) &= \mathcal{R}_0(z) + \kappa^2 \mathcal{R}_2(z) + \mathcal{O}(\kappa^3).
\end{aligned} \tag{3.1}$$

The tangential velocity v_2 is expected on physical grounds to be the dominant velocity component in fluid layer 2, and this is reflected in the form of the expansions (3.1). It turns out that the unknown constants A_1 and T_0 in Equations (2.2) and (2.4), corresponding to the reference density and the constant temperature in upper layer 1, respectively, must also be expanded (de-tuned) in powers of the Coriolis parameter κ, as for the dependent variables in Equations (3.1). These additional series take the form

$$\begin{aligned} A_1 &= A_{1,0} + \kappa^2 A_{1,2} + \mathcal{O}(\kappa^3), \\ T_0 &= T_{0,0} + \kappa^2 T_{0,2} + \mathcal{O}(\kappa^3). \end{aligned} \tag{3.2}$$

3.2. The zeroth-order equations

The perturbation series (3.1) and (3.2) are substituted in the full nonlinear system of Equations (2.5)–(2.14), and terms at each order of the expansion parameter κ are collected. At zeroth order, Equation (2.5) gives the approximate expression of mass conservation in the form

$$\frac{\partial(r\rho_{2,0}u_{2,0})}{\partial r} + \frac{\partial(r\rho_{2,0}w_{2,0})}{\partial z} = 0, \tag{3.3}$$

and the radial momentum equation (2.6) yields

$$-\frac{v_{2,0}^2}{r} + \frac{1}{\gamma\rho_{2,0}}\frac{\partial p_{2,0}}{\partial r} = 0. \tag{3.4}$$

The zeroth-order expression of conservation of momentum in the tangential direction is found from Equation (2.7) to be

$$u_{2,0}\frac{\partial v_{2,0}}{\partial r} + \frac{u_{2,0}v_{2,0}}{r} = 0, \tag{3.5}$$

and the vertical component of momentum conservation (Equation (2.8)), at this order, becomes

$$\frac{1}{\gamma\rho_{2,0}}\frac{\partial p_{2,0}}{\partial z} = -\frac{\varepsilon}{\gamma - 1}. \tag{3.6}$$

The energy equation (2.9) gives rise to the zeroth-order form

$$u_{2,0}\frac{\partial \mathcal{H}_{2,0}}{\partial r} + w_{2,0}\frac{\partial \mathcal{H}_{2,0}}{\partial z} = -\frac{\varepsilon w_{2,0}}{\gamma - 1}, \tag{3.7a}$$

in which the function $\mathcal{H}_{2,0}$ is defined to be

$$\mathcal{H}_{2,0} = \frac{p_{2,0}}{(\gamma - 1)\rho_{2,0}} + \tfrac{1}{2}v_{2,0}^2. \tag{3.7b}$$

The boundary conditions (2.10)–(2.14) are found to be obeyed by these zeroth-order functions.

4. Solution at zeroth order

We may find the essential structure of the cyclone vortex, as obtained with the present model, by solving the zeroth-order Equations (3.3)–(3.7), with accompanying boundary conditions (2.10)–(2.14). It turns out that the dependent variables can all be determined in closed form, and this convenient fact also enhances the qualitative understanding available as a result of the solution to this model.

We may solve the non-linear system of Equations (3.3)–(3.7) by recognizing that one possible solution to Equation (3.5) gives the zeroth-order tangential velocity component in the form

$$v_{2,0} = \frac{\Gamma}{r}, \tag{4.1}$$

where the constant Γ is proportional to the circulation produced by the vortex. Equations (3.4) and (3.6) may then be combined to yield a partial differential equation of first order for the pressure function $p_{2,0}$, from which it follows that the general solution for pressure and density at zeroth order is

$$\begin{aligned} p_{2,0}(r,z) &= h(\xi), \\ \rho_{2,0}(r,z) &= \frac{\gamma-1}{\gamma} h'(\xi), \quad \text{with} \quad \xi = D - \frac{(\gamma-1)\Gamma^2}{2r^2} - \varepsilon z. \end{aligned} \tag{4.2}$$

Up to this point, both the constant D and the function $h(\xi)$ remain arbitrary.

The form of the function h in Equations (4.2) must now be chosen in order to satisfy the energy Equation (3.7) at zeroth order. For arbitrary velocity component functions $u_{2,0}$ and $w_{2,0}$, it follows that the zeroth-order enthalpy function $\mathcal{H}_{2,0}$ in Equation (3.7b) must decay linearly with height z in layer 2. After a little analysis, it is found that the function h must take the form

$$h(\xi) = E\xi^{\gamma/(\gamma-1)}, \tag{4.3}$$

where the constant E is again arbitrary.

We determine the integration constants D and E in Equations (4.2) and (4.3) by applying the boundary conditions (2.13) far from the cyclone, and it is easily seen that $D = E = 1$ in these dimensionless variables. The zeroth-order solutions for pressure and density within the vortex in layer 2 therefore have the final form

$$\begin{aligned} p_{2,0}(r,z) &= \xi^{\gamma/(\gamma-1)}, \\ \rho_{2,0}(r,z) &= \xi^{1/(\gamma-1)}, \quad \text{with} \quad \xi = 1 - \frac{(\gamma-1)\Gamma^2}{2r^2} - \varepsilon z. \end{aligned} \tag{4.4}$$

We may compute the temperature within the cyclone from the ideal gas law, using Equations (4.4). In these non-dimensional variables,

$$T_{2,0} = \frac{p_{2,0}}{\rho_{2,0}} = 1 - \frac{(\gamma-1)\Gamma^2}{2r^2} - \varepsilon z, \tag{4.5}$$

which also obeys the requirement (2.13) at infinity. By applying the boundary conditions (2.13) and (2.14) and the matching condition (2.10) infinitely far from the cyclone, we can now determine the zeroth-order reference density $A_{1,0}$ and constant temperature $T_{0,0}$ in the isothermal portion of layer 1 (for $z > \delta$ in Equation (2.1)), in terms of the density jump $\Delta\rho$

at the interface, defined in Equation (2.14). After some calculation, these constants are found to be

$$T_{0,0} = \frac{(1-\varepsilon)^{\gamma/(\gamma-1)}}{(1-\varepsilon)^{1/(\gamma-1)} - \Delta\rho},$$
$$A_{1,0} = \frac{(1-\varepsilon)^{\gamma/(\gamma-1)}}{T_{0,0}} \exp\left(\frac{\varepsilon\gamma}{(\gamma-1)T_{0,0}}\right). \tag{4.6}$$

These results (4.6) follow from the fact that $\delta < 1$ for a mature cyclone, so that the matching condition (2.10) is applied in the isothermal region $z > \delta$, for $r \to \infty$.

A central feature of this model of a cyclone is the presence of a sharp interface between layers 1 and 2, lying along the surface $r = \mathcal{R}(z)$. This is a free boundary, and its (unknown) location is determined by the dynamics of the vortex, by means of the matching condition (2.10). At zeroth order in the expansions (3.1) and (3.2), this condition becomes

$$\left[1 - \frac{(\gamma-1)\Gamma^2}{2r^2} - \varepsilon z\right]^{\gamma/(\gamma-1)} = p_1(z) \quad \text{on} \quad r = \mathcal{R}(z),$$

in which the pressure $p_1(z)$ in layer 1 is given by the formula (2.4). This expression is combined with Equations (4.6), and a little manipulation yields the equations

$$\mathcal{R}_0(z) = \Gamma\left[\frac{(\gamma-1)/2}{1 - \varepsilon z - (1-\varepsilon)\exp(\varepsilon(1-\delta)/T_{0,0}) f(z)}\right]^{1/2} \quad 0 < z < \delta, \tag{4.7a}$$

in which the function

$$f(z) = \left[\frac{1}{T_{0,0}} - \frac{1 - T_{0,0}}{T_{0,0}}\frac{z}{\delta}\right]^{\varepsilon\delta/(1-T_{0,0})} \tag{4.7b}$$

has been defined for convenience, and

$$\mathcal{R}_0(z) = \Gamma\left[\frac{(\gamma-1)/2}{1 - \varepsilon z - (1-\varepsilon)\exp(\varepsilon(1-z)/T_{0,0})}\right]^{1/2} \quad z > \delta. \tag{4.7c}$$

These expressions enable the explicit determination of the interface shape $r = \mathcal{R}_0(z)$.

It follows from Equations (4.7) that the interface $r = \mathcal{R}_0(z)$ between stationary fluid layer 1 and the moving layer 2 always lies below the undisturbed level $z = 1$, and is ultimately drawn down to sea level by the vortex, to form the eye of the cyclone. This is formalized in the following theorem.

Theorem. *For* $\gamma > 1$ *and* $0 < \varepsilon < 1$ *and positive density jump* $\Delta\rho > 0$*, the interface always lies below the undisturbed level* $z = 1$ *for* $r < \infty$.

Proof. This result follows simply from the fact that the argument of the square root term in Equation (4.7c) must be positive. Therefore

$$1 - \varepsilon z - (1-\varepsilon)\exp(\varepsilon(1-z)/T_{0,0}) > 0.$$

By applying the real inequality $\exp(x) \geqslant 1 + x$ (see, for example, Abramowitz and Stegun [11, p. 70]), we can manipulate this expression to yield

$$[T_{0,0} - (1-\varepsilon)](1-z) > 0.$$

Now it follows from Equation (4.6) that $T_{0,0} > 1 - \varepsilon$ for positive density jump $\Delta\rho > 0$, and therefore $z < 1$ for every value of radius r, as required. This concludes the proof of the theorem.

A formula for the radius of the cyclone's eye is obtained easily from Equation (4.7a), using values $z = 0$ and $\mathcal{R}_0(0) = \alpha$. The result is

$$\alpha = \Gamma \left[\frac{(\gamma - 1)/2}{1 - (1-\varepsilon)\exp(\varepsilon(1-\delta)/T_{0,0}) f(0)} \right]^{1/2}, \tag{4.8}$$

where the function $f(z)$ is defined in Equation (4.7b). This equation shows that it is the presence of an interface in the atmosphere that causes an eye to form in a tropical cyclone, at least in the present model. From Equation (4.1), it follows that the tangential velocity at the eye, where $(r, z) = (\alpha, 0)$, has the value Γ/α, which from Equations (4.8) and (4.6) is independent of the storm strength Γ at this zeroth order of approximation.

So far, the radial and vertical velocity components $u_{2,0}$ and $w_{2,0}$ in the perturbation scheme (3.1) have not been determined, and the above solution (4.1)–(4.8) clearly allows considerable freedom in their choice. We could, of course, set these to zero without altering the above results for pressure, density or interface location, but in practice it is known that secondary inflow and outflow occurs within cyclones (see, for example, Lutgens and Tarbuck [1]), and so it would be desirable to reproduce this behaviour in the present model. To this end, it is sufficient to observe that the zeroth-order Equation (3.3) of mass conservation is satisfied identically by a streamfunction $\Psi(r, z)$ having the property that

$$u_{2,0} = \frac{1}{r\rho_{2,0}} \frac{\partial \Psi}{\partial z} \qquad w_{2,0} = \frac{1}{r\rho_{2,0}} \frac{\partial \Psi}{\partial r}. \tag{4.9}$$

The kinematic boundary condition (2.11) on the free surface $r = \mathcal{R}_0(z)$ and the bottom condition (2.12) then require that the stream function Ψ be constant on both these surfaces; since these surfaces intersect along the circle $(r, z) = (\alpha, 0)$, the constant value of the stream function must be the same on both surfaces, and so we set

$$\Psi = 0 \quad \text{on } z = 0 \quad \text{and } r = \mathcal{R}_0(z). \tag{4.10}$$

A stream function Ψ is sought, satisfying the condition (4.10), such that there will be an inflow towards the cyclone near the sea surface $z = 0$, and an outflow from the cyclone high in the atmosphere, along the interface $r = \mathcal{R}_0(z)$. An example of a function meeting these criteria is the stream function

$$\Psi(r, z) = Mz[z - G(r, z)], \tag{4.11}$$

in which M is a constant and the function $G(r, z)$ is

$$G(r, z) = \frac{1}{\varepsilon} \left[1 - \frac{(\gamma - 1)\Gamma^2}{2r^2} - (1-\varepsilon)\exp(\varepsilon(1-\delta)/T_{0,0}) f(z) \right] \quad 0 < z < \delta, \tag{4.12a}$$

where the function $f(z)$ is as defined in Equation (4.7b), and

$$G(r,z) = \frac{1}{\varepsilon}\left[1 - \frac{(\gamma-1)\Gamma^2}{2r^2} - (1-\varepsilon)\exp(\varepsilon(1-z)/T_{0,0})\right] \quad z > \delta. \tag{4.12b}$$

We may compute the velocity components for the secondary radial and vertical flow in closed form, using Equations (4.9) and (4.12).

5. Numerical examples

In this section, the zeroth-order solution obtained in Section 4 is briefly illustrated with some numerical calculations, in order to highlight the features of the cyclone predicted by the present model.

As indicated in Section 2, the temperature high in the isothermal portion of the atmosphere is approximately $-70°$C, so that for a sea temperature of about $30°$C, the dimensionless parameter $T_{0,0}$ is expected to be about 0·67. Equation (4.6) then indicates that the parameter ε should have a value of about $\varepsilon = 0{\cdot}5$, to be consistent with this dimensionless temperature parameter $T_{0,0}$. In dimensional terms, this corresponds to a height of about $H = 16$ kms for the top of the cyclone, and this is consistent with observation (see, for example, Tarakanov [10] and ref. [2]). As the atmosphere becomes isothermal at about 11 kms height, the parameter δ should be chosen to be about $\delta = 0{\cdot}7$.

Figure 2 shows three interface shapes, computed at zeroth order from Equations (4.7), for parameter values $\gamma = 1{\cdot}4, \varepsilon = 0{\cdot}48, \delta = 0{\cdot}7$ and density jump $\Delta\rho = 0{\cdot}035$. The circulation had the values $\Gamma = 0{\cdot}2, 0{\cdot}5$ and 1, respectively, as indicated on the diagram. For these parameter values, the temperature in the upper layer 1 was computed from Equation (4.6) to be $T_{0,0} = 0{\cdot}63$ (recall that the sea-surface temperature has been normalized to 1). As the circulation Γ in the cyclone is increased, the interface moves closer to sea level and the radius of the eye increases, in accordance with the formula (4.8). Notice that the interface between layers 1 and 2 is very nearly vertical for $\Gamma = 0{\cdot}2$, and developes a slight over-hanging portion as the circulation Γ is increased. The eye radii for the three values of circulation $\Gamma = 0{\cdot}2, 0{\cdot}5$ and 1 have been computed from Equation (4.8) to be 0·98, 2·46 and 4·92 respectively, and for a cyclone of height $H = 15$ kms, these correspond to dimensional values of about 14·8 kms, 36·9 kms and 73·9 kms respectively. At the eye, the wind speed at the zeroth order is $\Gamma/\alpha = 0{\cdot}203$, which corresponds to a dimensional speed of about 255 kms/hr. This is consistent with measured values; Tarakanov [10, p. 159] indicates that speeds as high as 90 m/sec (324 kms/hr) have been measured, and in Australia, the highest wind speed recorded over land was 259 kms/hr (see ref. [2]).

Contour plots are shown in Figures 3(a), (b) and (c), for the pressure, density and temperature respectively, in the case when $\gamma = 1{\cdot}4, \varepsilon = 0{\cdot}48, \delta = 0{\cdot}7$ and $\Delta\rho = 0{\cdot}035$, as above, and the circulation has the value $\Gamma = 0{\cdot}5$. (Here, the eye has radius $\alpha = 2{\cdot}46$). The background pressure $p_1(z)$ for a stationary atmosphere, given by Equation (2.4), has been subtracted from the total pressure to create the contour plot in Figure 3(a). This enables the pressure deviation caused by the vortex to be seen more clearly, although the total pressure is, of course, continuous cross the eye wall. By contrast, the density has a step discontinuity across the interface, as is evident in Figure 3(b), where the contour lines may exhibit an abrupt jump in this region.

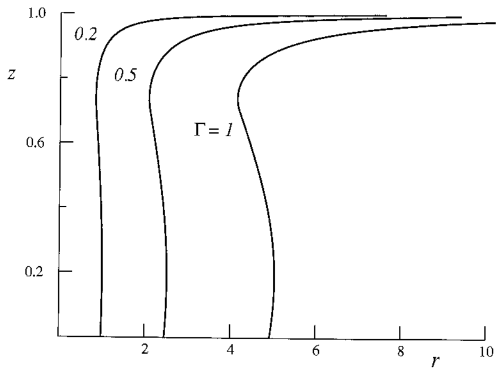

Figure 2. Three interface shapes computed with circulation $\Gamma = 0{\cdot}2, 0{\cdot}5$ and 1. The ratio of specific heats, compressibility parameter and density jump have the values $\gamma = 1{\cdot}4$, $\varepsilon = 0{\cdot}48$ and $\Delta\rho = 0{\cdot}035$. The dimensionless height of the isothermal portion of the atmosphere is $\delta = 0{\cdot}7$.

The temperature, too, may be discontinuous at the interface, as may be seen from Figure 3(c). Contours have been shown for 20 equal increments in temperature, from the (dimensionless) value $T = 1$ at the sea surface, to the isothermal value $T = T_{0,0}$ high in the atmosphere. The equally spaced contours within the eye indicate the linear decrease in temperature in this region, consistent with Equation (2.1), and the isothermal region is also clearly visible. Notice that the temperature within the eye, close to the sea level, is slightly higher than the temperature within the cyclone, and this is consistent with Equation (4.5) for small z.

In order to demonstrate that inflow and outflow can be accommodated within the present model, the test velocities in Equations (4.9)–(4.12) have been plotted in vector form in Figure 4, for the same parameter values as in Figures 3, and assuming a dimensionless coriolis parameter $\kappa = 4{\cdot}3 \times 10^{-3}$. The interface $r = \mathcal{R}_0(z)$ is also shown, to highlight the lower fluid in layer 2. Although the numerical values for the velocity components $\kappa u_{2,0}$ and $\kappa w_{2,0}$ in Equation (4.9) are small (with $M = 1$), Figure 4 nevertheless shows that it is possible to create stream functions Ψ in Equation (4.11) which can generate credible secondary flows within the cyclone.

6. Relationship to tropical cyclones

As discussed in Section 1, the aim of this note is to demonstrate that a relatively simple model of a tropical cyclone can reproduce many of the observed features of the phenomenon, using an approximate perturbation theory that is sufficiently straightforward as to remain analytically tractable. The central feature of this model is the presence of an interface in the atmosphere, separating an outer rotating fluid layer from an inner stationary layer of different density. The location of the interface is to be determined in this model, and to zeroth order it is given by Equation (4.7). The eye radius α is given in closed form by Equation (4.8) to this order of approximation. The radius α increases linearly with the vortex strength Γ, which therefore could be estimated from measurements of the eye radius.

This simple model is capable of reproducing at least the gross features of actual tropical cyclones. When the temperature difference between the air at sea level and high in the atmosphere is taken into account (by means of the dimensionless ratio $T_{0,0}$), Equation (4.6) reveals that the parameter ε must be about $0{\cdot}5$, corresponding to a cyclone height near the eye

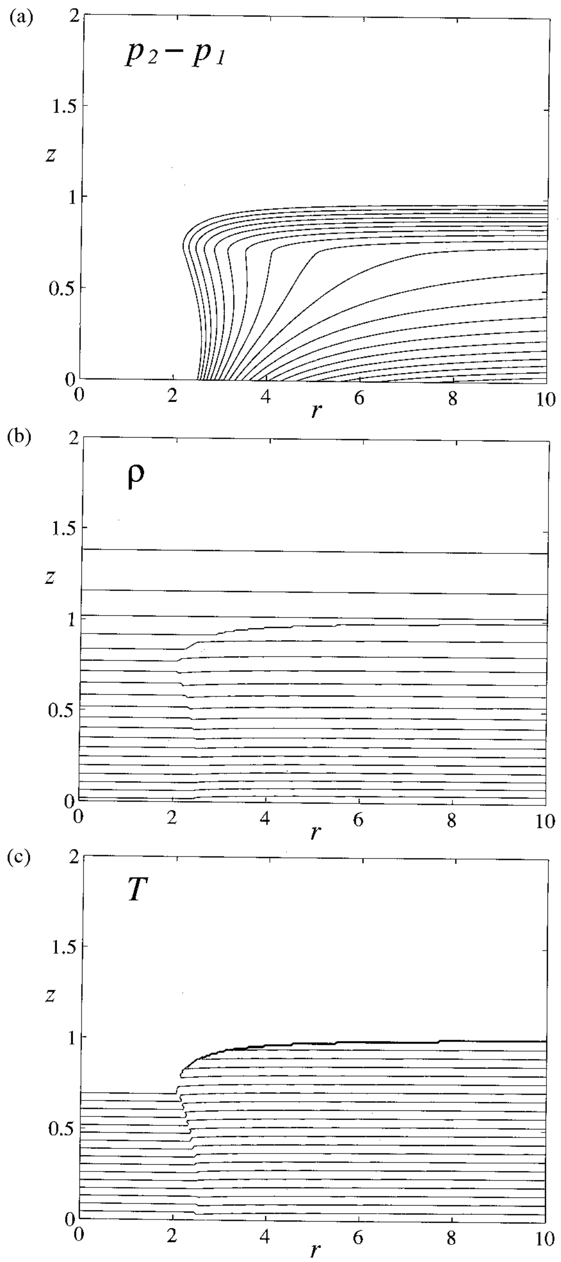

Figure 3. (a) Pressure difference, (b) Density and (c) Temperature contours for a cyclone in which $\gamma = 1{\cdot}4$, $\varepsilon = 0{\cdot}48$, $\Delta\rho = 0{\cdot}035$, $\delta = 0{\cdot}7$ and circulation $\Gamma = 0{\cdot}5$.

of about $H = 15$ kms. This is consistent with measured values, and indicates that the model is crudely correct in its description of the energy balance within the cyclone (heat energy gained at sea level is responsible for the circulation Γ). As a consequence, physically acceptable values for wind speeds near the eye may be obtained with this model. It is also possible to generate credible secondary flows within the cyclone, and an example of such a flow has been given. In a more complete model of the cyclone, the arbitrariness of these secondary flows

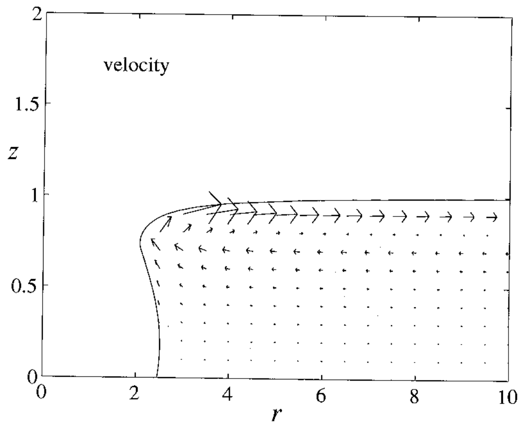

Figure 4. Secondary flow within the cyclone, for the case $\gamma = 1{\cdot}4$, $\varepsilon = 0{\cdot}48$, $\Delta\rho = 0{\cdot}035$, $\delta = 0{\cdot}7$ and $\Gamma = 0{\cdot}5$. The Coriolis parameter is $\kappa = 4{\cdot}3 \times 10^{-3}$, and the interface is also shown on the diagram.

would be removed through a precise statement of the nature of the conditions prevailing near ocean boundary layers, and so on, but simple analytical formulae would then be unlikely in such a model. If desired, we could perhaps tune the simple model of the present paper further by allowing the dimensionless density jump $\Delta\rho$ in Equation (2.14) to be a weakly varying function of height z, similar to the expressions in the perturbation scheme (3.1).

The present steady model of a tropical cyclone has assumed the linear temperature profile (2.1) within the eye, in the region $0 < z < \delta$ closest to the sea surface. The details of the time-dependent cyclone start-up process and evolution have therefore not been considered here, but it is clear that considerable air exchange across the interface must have taken place, at least during this initial phase. This is because it would not be possible for all the air within the eye of the mature cyclone to have descended from the upper layer, initially at a height of approximately 15 kms, since descent of this air under adiabatic conditions would raise its temperature to about 110°C, which is far above the sea-surface temperature of about 30°C. Thus air exchange across the interface must have occurred. In our model, this is accounted for implicitly by the assumed temperature profile (2.1) within the eye, although Simpson and Riehl [12, p. 147] suggest that transport across the eye wall continues even in the mature tropical cyclone. We could incorporate this additional feature into the formulation given here, by replacing Equation (2.11) with a more realistic model of fluid exchange, although more attention would then need to be paid to the details of air movement within the eye. It is highly unlikely that the inclusion of such effects would permit a closed form solution of the type presented here.

7. Concluding remarks

In this note, we have discussed a simple analytical model of a tropical cyclone, and have obtained a simple qualitative explanation for the structure of this phenomenon. In addition, analytical formulae relating the flow variables within the cyclone have been derived, without the need for empirical data fitting. The central feature of our model is the presence of a sharp interface between two air layers of different densities. The rotation of the vortex causes the interface to be drawn down to sea level $z = 0$, in a steadily circulating flow, and this is revealed to be the primary explanation for eye formation within a cyclone. In this sense, the cyclone is behaving in a manner vaguely similar to the familiar bath-plug vortex [13], at least at this level of approximation.

It must, of course, be admitted that the simple understanding gained from a model such as this is obtained at the expense of accuracy, so that the predictions made here are only qualitative. In particular, the r^{-1} tangential velocity profile (4.1) obtained here is too simplistic, and experimental evidence suggests a slower decay in wind speed with distance, perhaps of the form r^{-n}, typically with $0{\cdot}4 < n < 0{\cdot}6$ (see ref. [7]). Indeed, Holland [7] makes use of a rather complicated exponential formula, obtained from empirical data fitting. We have assumed radial symmetry and the presence of a sharp interface between the two fluid layers, so ignoring the asymmetry of the cyclone and the need for a more realistic atmosphere model. In addition, boundary-layer friction has been ignored along with the effects of moisture. No attempt has been made to allow for air exchange across the eye wall, or changes to the tangential velocity component v_2 with height z in the vortex. Accounting for these phenomena requires a fully numerical solution of the governing equations, with a consequent increase in the difficulty of comprehension. The present analysis may serve as a starting solution in such a numerical model.

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